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J. Math. Anal. Appl. 286 (2003) 682–691

Journal of
 MATHEMATICAL
 ANALYSIS AND
 APPLICATIONS

www.elsevier.com/locate/jmaa

Nonlinear higher order boundary value problems with multiple positive solutions

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Received 30 January 2003

Submitted by D. O'Regan

1. Introduction

We consider nonlinear boundary value problems of the form

$$y^{(n)} + f(y) = 0, \quad 0 \leq x \leq 1, \quad (1)$$

$$y^{(k)}(0) = 0, \quad \text{for } k = 0, 1, \dots, n-2, \quad (2)$$

$$y^{(j)}(1) = 0, \quad \text{for one fixed } j \text{ satisfying } 1 \leq j \leq n-2, \quad (3)$$

where $f(y) \geq 0$, $n \geq 2$, and $f(y)$ is continuous for $y \in \mathbb{R}$. For given n , j , and N , we formulate conditions on f which guarantee the existence of at least N positive solutions. Our motivation is the recent work of Henderson and Thompson [7] which obtains such a result in the cases where $N = 3$, $j = 0$ or $j = n-2$ using the Leggett–Williams fixed point theorem. Their theorem in the case $j = 0$ essentially reads as follows.

Theorem 1. *Suppose $f : \mathbb{R} \rightarrow [0, \infty)$ is continuous. Then there exist numbers K , L , q so that if $0 < a < b < c/q$ and f satisfies*

$$f(y) < La \quad \text{for } 0 \leq y \leq a,$$

$$f(y) \geq Kb \quad \text{for } b \leq y \leq qb,$$

$$f(y) \leq Lc \quad \text{for } 0 \leq y \leq c,$$

then the boundary value problem consisting of (1), (2), and $y(1) = 0$ has at least three nonnegative solutions, at least two of which are strictly positive on $(0, 1)$.

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The value of q in [7] is 4^{n-1} ; the values of K , L involve the computation of certain integrals of the Green's function for the problem $-y^{(n)} = 0$ and the boundary conditions as given. A similar theorem is given in [7] for the problem consisting of (1), (2), and $y^{(n-2)}(1) = 0$, but is not correct as stated because of an unfortunate error which the authors have now corrected.

Our purpose here is to provide results which cover all the cases included in (3). In case $j = n - 2$, our numbers K , L , and q may be the same as the corrected statement Henderson and Thompson now have; they are the same as those reported privately by Thompson for small values of n . Our results also include the cases $0 < j < n - 2$, for $n \geq 4$.

Recent results on symmetric even order problems, beginning with [6] and continuing with [1–3,5] have used shooting methods [2,3] and a fixed point theorem of Krasnosel'skii [1,5]. Here, we also use shooting methods, but make no use of Green's functions. We obtain explicit formulas for our numbers K , L , q . Our results also include the general case for N not necessarily 3. Much of the work on problems with multiple solutions over the last twenty years was motivated by the paper of Parter [8]. A related paper which seems to have escaped the notice of recent authors is that by Brown et al. [4].

Since we use initial value methods, we shall assume that f is Lipschitz continuous so that we may use the standard theorems on uniqueness in initial value problems and continuous dependence on initial conditions. We use the familiar notation

$$\|y\| = \sup\{|y(x)| : 0 \leq x \leq 1\}.$$

2. Preliminary results

Since the second order case reduces to the case considered in essentially all of the references mentioned earlier, we will fix the integer n and assume that $n \geq 3$. We also fix j with $1 \leq j \leq n - 2$.

Our work will be based on solutions y_m of the initial value problem IVP(m) consisting of (1) and the initial conditions

$$y^{(k)}(0) = 0, \quad \text{for } k = 0, 1, \dots, n - 2, \quad y^{(n-1)}(0) = m. \quad (4)$$

We need three preparatory lemmas.

Lemma 1. Suppose that $f : \mathbb{R} \rightarrow [0, Q]$ is Lipschitz continuous. Then the solution of IVP(m) exists on the entire interval $[0, 1]$ and satisfies

$$\frac{mx^{n-k-1}}{(n-k-1)!} \geq y_m^{(k)}(x) \geq \frac{mx^{n-k-1}}{(n-k-1)!} - \frac{Qx^{n-k}}{(n-k)!}, \quad 0 \leq x \leq 1,$$

for $0 \leq k \leq n - 1$.

Proof. Let y_m be the solution, guaranteed by the standard local existence theorem, of the initial value problem IVP(m). If this solution does not exist on the entire interval $[0, 1]$, we let $[0, d)$ be the maximum interval of existence. Thus

$$0 \leq -y_m^{(n)}(x) = f(y_m(x)) \leq Q, \quad 0 \leq x < d.$$

Integrating from 0 to $x \in (0, d)$ gives

$$m \geq y_m^{(n-1)}(x) \geq m - Qx,$$

and integrating again

$$mx \geq y_m^{(n-2)}(x) \geq mx - \frac{Qx^2}{2}.$$

Repeated integration leads to

$$\frac{mx^{n-k-1}}{(n-k-1)!} \geq y_m^{(k)}(x) \geq \frac{mx^{n-k-1}}{(n-k-1)!} - \frac{Qx^{n-k}}{(n-k)!}, \quad 0 \leq x < d,$$

for $0 \leq k \leq n-1$. These bounds show that y_m exists on the entire interval $[0, 1]$, so $d = 1$ and the desired bounds hold on $0 \leq x \leq 1$. \square

Lemma 2. Let $a < d$ and $\ell \geq 2$. Suppose $w^{(\ell)}(x) \leq 0$ for $a \leq x \leq d$, $w^{(k)}(a) \geq 0$ for $0 \leq k \leq \ell-2$, and $w^{(\ell-1)}(a) > 0$. If $w(d) \geq 0$, then $w(x) > 0$ on (a, d) .

Proof. Since $w^{(\ell-1)}(a) > 0$, there exists an $\epsilon > 0$ such that $w^{(\ell-1)}(x) > 0$ on $(a, a + \epsilon)$. Thus, for $a < x \leq a + \epsilon$, we integrate to get

$$w^{(\ell-2)}(x) = w^{(\ell-2)}(a) + \int_a^x w^{(\ell-1)}(t) dt > 0,$$

and continuing to integrate,

$$w(x) = w(a) + \int_a^x w'(t) dt > 0, \quad \text{for } a < x \leq a + \epsilon.$$

If $w(x) > 0$ on (a, d) is false, then $w(c_0) = 0$ for some point $c_0 \in (a, d)$. So $w(a) \geq 0$, $w(c_0) = 0$, and $w(d) \geq 0$. If $\ell \geq 3$, there exists $a_1 \in (a, c_0)$, $d_1 \in (c_0, d)$ such that $w'(a_1) \leq 0$ and $w'(d_1) \geq 0$. So $w'(a) \geq 0$, $w'(a_1) \leq 0$, and $w'(d_1) \geq 0$. Continuing by induction, we obtain $w^{(\ell-2)}(a) \geq 0$, $w^{(\ell-2)}(a_{\ell-2}) \leq 0$, and $w^{(\ell-2)}(d_{\ell-2}) \geq 0$, where $a < a_{\ell-2} < d_{\ell-2} \leq d$. We now show that $w^{(\ell-2)} \equiv 0$ on $[a, a_{\ell-2}]$, contradicting the fact that $w^{(\ell-1)}(a) > 0$. Arguing by contradiction, there is a point $\delta \in (a, a_{\ell-2})$ where $w^{(\ell-2)}(\delta) \neq 0$. If $w^{(\ell-2)}(\delta) < 0$, then there exist a $\delta_1 \in (a, \delta)$ and a $\delta_2 \in (\delta, a_{\ell-2})$ so that $w^{(\ell-1)}(\delta_1) < 0$ and $w^{(\ell-1)}(\delta_2) > 0$. But then there exists a $\delta_3 \in (\delta_1, \delta_2)$ where $w^{(\ell)}(\delta_3) > 0$, contradicting $w^{(\ell)}(x) \leq 0$. If $w^{(\ell-2)}(\delta) > 0$, then there exists a $\delta_1 \in (\delta, a_{\ell-2})$ where $w^{(\ell-1)}(\delta_1) < 0$ and a $\delta_2 \in (a_{\ell-2}, d_{\ell-2})$ so that $w^{(\ell-1)}(\delta_2) \geq 0$. Again there exists a $\delta_3 \in (\delta_1, \delta_2)$ where $w^{(\ell)}(\delta_3) > 0$, contradicting $w^{(\ell)}(x) \leq 0$. Thus, $w(x) > 0$ on (a, d) . \square

Lemma 3. Let $a < d$ and $\ell \geq 2$. Suppose $w^{(\ell)}(x) \leq 0$, $w^{(\ell)}(x) \not\equiv 0$ for $a \leq x \leq d$, and $w^{(k)}(a) = 0$ for $0 \leq k \leq \ell-2$. If $w(d) \geq 0$, then $w(x) > 0$ on (a, d) .

Proof. It suffices to show that these hypotheses imply the hypotheses of Lemma 2; we need only show that $w^{(\ell-1)}(a) > 0$. Supposing the contrary, $w^{(\ell-1)}(a) \leq 0$. Then by our hypotheses,

$$w^{(\ell-1)}(x) = w^{(\ell-1)}(a) + \int_a^x w^{(\ell)}(t) dt \leq 0, \quad \text{if } a \leq x \leq d,$$

and $w^{(\ell)}(d) < 0$. Continuing by induction, we arrive at

$$w(x) = w(a) + \int_a^x w'(t) dt < 0, \quad \text{if } a \leq x \leq d,$$

and $w(d) < 0$, contradicting our assumption that $w(d) \geq 0$. \square

Lemmas 4 and 5 below give our values of L , K , and q ; these values depend on n and j .

Lemma 4. Let $L = L_{n,j}$ where $L_{n,j} = (n-j)n!/j$. Suppose that $f : [0, c] \rightarrow [0, \infty)$ is Lipschitz continuous and satisfies $f(y) \leq Lc$, $f(y) \not\equiv Lc$, for $0 \leq y \leq c$. Then there exist numbers $0 \leq s_1 < m^*$ for which $0 \leq y_{s_1} < c$ solves the boundary value problem (1)–(3) and $y_{m^*} < c$ exists on the interval $[0, 1]$. Moreover, if $m > s_1$ and $\|y_m\| < c$, then $y_m^{(j)}(1) > 0$.

Proof. We modify our differential equation (1) by temporarily defining f outside the interval $[0, c]$ by setting $f(y) = f(c)$ for $y > c$ and $f(y) = f(0)$ for $y < 0$. Then $0 \leq f(y) \leq Q = Lc$, for $y \in \mathbb{R}$. Using the bound of Lemma 1 at $x = 1$, we obtain for $k = j$

$$\frac{m}{(n-j-1)!} \geq y_m^{(j)}(1) \geq \frac{m}{(n-j-1)!} - \frac{Lc}{(n-j)!}.$$

Notice that if $m = 0$ then $y_m^{(j)}(1) \leq 0$ and if $m > Lc/(n-j)$ then $y_m^{(j)}(1) > 0$. The standard theorem on continuous dependence implies the existence of m with $0 \leq m \leq Lc/(n-j)$ so that y_m is a solution of (1)–(3) with the modified $f(y)$. We choose s_1 to be the largest such m . Then continuous dependence guarantees that $y_m^{(j)}(1) > 0$ for $m > s_1$, verifying the final assertion of the lemma. We now show that $0 \leq y(x) < c$ for $0 \leq x \leq 1$, so that y_{s_1} solves the unmodified problem. Suppose that the desired inequality is not true. It is easy to see that the solution v of $v^{(n)} + Lc = 0$ satisfying the boundary conditions

$$v^{(k)}(0) = 0, \quad \text{for } k = 0, 1, \dots, n-2, \quad \text{and} \quad v^{(j)}(1) = 0$$

is

$$v(x) = \frac{-Lcx^n}{n!} + \frac{Lcx^{n-1}}{(n-j)(n-1)!}.$$

Note that $v'(x) > 0$ on $(0, 1)$ and our choice of L implies $v(x)$ has its maximum value $v(1) = c$.

Let

$$u(x) = v(x) - y_{s_1}(x).$$

Now $u^{(n)}(x) = v^{(n)}(x) - f(y_{s_1}(x)) = -Lc + f(y_{s_1}(x)) \leq 0$. Moreover, $u^{(n)}(x) \neq 0$ because the range of y_{s_1} contains $[0, c]$. Thus Lemma 2, with $w = u^{(j)}$ and $\ell = n - j$, implies $u^{(j)}(x) > 0$ on $(0, 1)$ and therefore

$$y_{s_1}^{(j)}(x) < v^{(j)}(x), \quad \text{for } 0 < x < 1.$$

Integrating repeatedly, if necessary, from 0 to $x \in (0, 1]$ gives

$$y_{s_1}(x) < v(x) \leq c,$$

a contradiction. Using continuous dependence again, we may choose $m^* > s_1$ sufficiently close so that $y_{m^*} < c$ and $y_{m^*}^{(j)}(1) > 0$. Thus y_{s_1} and y_{m^*} are solutions of the unmodified equation (1), completing the proof of Lemma 4. \square

In order to state the following lemma, which contains values for K and q , we need some notation. Let $\alpha_n = 2^{n-1}(n-1)!$ and

$$\beta_n(k) = \frac{2^k(n-1)!}{(n-k-1)!}.$$

We also let

$$h(x) = \sum_{i=2}^n \frac{\beta_n(n-i)(x-1/2)^{n-i}}{(n-i)!} + \frac{\alpha_n(x-1/2)^{n-1}}{(n-1)!} - \frac{K(x-1/2)^n}{n!}.$$

Lemma 5. Let $1 \leq j \leq n-2$ and let $K = K_{n,j}$, where

$$K_{n,j} = \sum_{i=2}^{n-j} \frac{\beta_n(n-i)2^i(n-j)!}{(n-j-1)!} + 2(n-j)\alpha_n,$$

and let $q = q_{n,j}$, where

$$q_{n,j} = h(1) = \sum_{i=2}^n \frac{\beta_n(n-i)}{2^{n-i}(n-i)!} + \frac{\alpha_n}{2^{n-1}(n-1)!} - \frac{K}{2^n n!}.$$

Suppose that $b > 0$ and $f: [0, qb] \rightarrow [0, \infty)$ is Lipschitz continuous and satisfies $f(y) \geq Kb$ for $0 < b \leq y \leq qb$. Then there exists $m' > 0$ for which $\|y_{m'}\| \leq qb$,

$$y_{m'}^{(j)}(1) < 0, \quad \max\{y_{m'}(x): 0 \leq x \leq 1/2\} = b,$$

and

$$\max\{y_m(x): 0 \leq x \leq 1/2\} > b, \quad \text{for all } m > m'.$$

Proof. Extend f to all of \mathbb{R} by defining $f(y) = f(qb)$ for $y > qb$ and $f(y) = f(0)$ for $y < 0$. We first show that there exists $M > 0$ so that

$$\max\{y_m(x): 0 \leq x \leq 1/2\} > b, \quad \text{for } m > M. \quad (5)$$

Suppose otherwise; then there exist arbitrarily large values of m for which

$$\max\{y_m(x): 0 \leq x \leq 1/2\} \leq b.$$

For such m , we have

$$-y_m^{(n)}(x) = f(y_m(x)) \leq Q \equiv \max\{f(y): 0 \leq y \leq b\}, \quad 0 \leq x \leq 1/2.$$

Integrating, as in the proof of Lemma 1, we get

$$y_m(x) \geq \frac{mx^{n-1}}{(n-1)!} - \frac{Qx^n}{n!}.$$

It follows that

$$y_m(1/2) \geq \frac{1}{2^{n-1}(n-1)!} \left[m - \frac{Q}{2n} \right] > b,$$

if

$$m > M \equiv \frac{Q}{2n} + 2^{n-1}(n-1)!b.$$

Thus the set of all M satisfying (5) is not empty and this set is clearly bounded below by 0; let m' be the infimum of this set. Continuous dependence then implies that

$$\max\{y_{m'}(x): 0 \leq x \leq 1/2\} = b.$$

Let c be the point in $[0, 1/2]$ where this maximum occurs. We consider first the possibility that $0 < c < 1/2$. In this case, we may use the mean value theorem repeatedly if necessary to find $d \in (0, c]$ for which $y_{m'}^{(j)}(d) = 0$. Then $y_{m'}^{(j)}(1) < 0$, since otherwise Lemma 2, with $w = y_{m'}^{(j)}$, $\ell = n - j$, and $[a, d] = [0, 1]$, gives a contradiction. Moreover, $y_{m'}(c) = b$ is the absolute maximum of $y_{m'}$ on $[0, 1]$, for c is the only critical point of $y_{m'}$ in $[c, 1)$ since the existence of a critical point $c' \in (c, 1)$ would contradict Lemma 2 with $w = y_{m'}'$, $\ell = n - 1$, and $[a, d] = [0, c']$. Thus $\|y_{m'}\| = b < qb$, completing the proof in this case.

If, on the other hand, $c = 1/2$, we proceed as follows and first show that $y_{m'}^{(n-1)}(1/2) < \alpha_n b$. Supposing otherwise, we get $y_{m'}^{(n-1)}(1/2) \geq \alpha_n b$. Then $y_{m'}^{(n)}(1/2) = -f(b) < 0$ implies that $y_{m'}^{(n-1)}(x) > \alpha_n b$ for $0 \leq x < 1/2$. Integrating $n - 1$ times gives

$$y_{m'}(x) > \frac{\alpha_n b x^{n-1}}{(n-1)!}$$

for $0 < x \leq 1/2$ and thus

$$y_{m'}(1/2) > \frac{\alpha_n b}{2^{n-1}(n-1)!} = b,$$

contradicting the definition of m' . Next we get bounds on lower derivatives of $y_{m'}$ at $x = 1/2$. Since $y_{m'}^{(n)} \leq 0$ then $y_{m'}^{(n-2)}$ is concave down. Thus, estimating the integral with an inscribed triangle,

$$\frac{y_{m'}^{(n-2)}(x)}{2} x \leq \int_0^x y_{m'}^{(n-2)}(t) dt = y_{m'}^{(n-3)}(x) - y_{m'}^{(n-3)}(0)$$

and therefore

$$x y_{m'}^{(n-2)}(x) \leq 2 y_{m'}^{(n-3)}(x).$$

Integrating repeatedly, we get for $k = 2, 3, \dots, n-1$

$$x y_{m'}^{(n-k)}(x) \leq k y_{m'}^{(n-k-1)}(x).$$

Iterating this last inequality for $x = 1/2$, we get for $1 \leq k \leq n-2$

$$y_{m'}^{(k)}(1/2) \leq \frac{2^k (n-1)!}{(n-k-1)!} y_{m'}(1/2) = \beta_n(k)b. \quad (6)$$

We next prove an auxiliary result which we will need to use twice. Suppose $1 \leq i \leq n-2$ and that $y_{m'}^{(i)}(d) \geq 0$, for some $d \in (1/2, 1]$. Then Lemma 2, with $w = y_{m'}^{(i)}$ and $\ell = n-i$, tells us that $y_{m'}^{(i)}(x) > 0$ on $(0, d)$. Integrating this inequality repeatedly if necessary leads to $y_{m'}'(x) > 0$ on $(0, d)$. Thus $y_{m'}$ is increasing on $(0, d)$ and $y_{m'}(x) > b$ on $(1/2, d]$. Let

$$\delta = \sup\{x \in [1/2, d]: y_{m'}(x) \leq qb\}.$$

Then

$$y_{m'}^{(n)}(x) = -f(y_{m'}(x)) \leq -Kb \quad \text{for } 1/2 \leq x \leq \delta.$$

Integrating this last inequality over the interval $[1/2, x] \subset [1/2, \delta]$ and using our bound on $y_{m'}^{(n-1)}(1/2)$, we get

$$y_{m'}^{n-1}(x) < [\alpha_n - Kb(x-1/2)]b. \quad (7)$$

Integrating this last inequality repeatedly over the interval $[1/2, x] \subset [1/2, \delta]$ and using the inequalities (6), we obtain for $0 \leq k \leq n-2$

$$y_{m'}^{(k)}(x) < \left[\sum_{i=2}^{n-k} \frac{\beta_n(n-i)(x-1/2)^{n-k-i}}{(n-k-i)!} + \frac{\alpha_n(x-1/2)^{n-k-1}}{(n-k-1)!} - \frac{K(x-1/2)^{n-k}}{(n-k)!} \right] b. \quad (8)$$

When $0 \leq k \leq n-2$, the right side of (8) is $h^{(k)}(x)b$; the right side of (7) is $h^{(n-1)}(x)b$ and $h^{(n)}(x) \equiv -Kb$. Then by construction, $h^{(k)}(1/2) = \beta_n(k) > 0$ for $0 \leq k \leq n-2$ and $h^{(n-1)}(1/2) = \alpha_n > 0$. Our value of K is chosen so that $h^{(j)}(1) = 0$. By Lemma 3, with $w = h^{(j)}$ and $\ell = n-j$, $h^{(j)}(x) > 0$ on $(1/2, 1)$. Integrating if necessary, we see that $h'(x)$ is positive on $(1/2, 1)$ so $h(x)$ is strictly increasing. We conclude that

$$y_{m'}(\delta) < h(\delta)b \leq h(1)b = qb.$$

The definition of δ implies $\delta = d$ and thus

$$y_{m'}(x) \leq qb \quad \text{for } 0 \leq x \leq d. \quad (9)$$

We now prove $y_{m'}^{(j)}(1) < 0$. Supposing the contrary, $y_{m'}^{(j)}(1) \geq 0$ and we may apply the result of the previous paragraph with $i = j$, $d = 1$. Thus we may use (8) to conclude that $y_{m'}^{(j)}(1) < h^{(j)}(1)b = 0$, a contradiction. To complete the proof, we need to show that $\|y_{m'}\| \leq qb$. Let d be the point in $(0, 1]$ at which $y_{m'}$ attains its absolute maximum. Then

$d \in [1/2, 1]$ and $y_{m'}(d) \geq 0$. If $d = 1/2$, the result is trivially true. Otherwise, we may apply the result of the previous paragraph with $i = 1$ and then (9) implies the desired conclusion. \square

3. Main results

Now we can state our main theorem.

Theorem 2. Let L be as in Lemma 4 and q, K as in Lemma 5. Suppose $k \geq 0$ is an integer and $0 < c_0 < b_1 < c_1/q < b_2/q < c_2/q^2 < \dots < b_k/q^{k-1} < c_k/q^k$, and that $f : [0, c_k] \rightarrow [0, Lc_k]$ is Lipschitz continuous and satisfies

$$\begin{aligned} f(y) &\leq Lc_0, & f(y) &\neq Lc_0, & \text{for } 0 \leq y \leq c_0; \\ f(y) &\geq Kb_i, & \text{for } b_i \leq y \leq qb_i & \text{and } i = 1, 2, \dots, k; \\ f(y) &\leq Lc_i, & \text{for } 0 \leq y \leq c_i & \text{and } i = 1, 2, \dots, k. \end{aligned}$$

Then the boundary value problem (1)–(3) has $2k + 1$ nonnegative nondecreasing solutions $y_1, y_2, \dots, y_{2k+1}$, with $0 \leq y_1^{(n-1)}(0) < y_2^{(n-1)}(0) < \dots < y_{2k+1}^{(n-1)}(0)$. Moreover, $y_i(x)$ is strictly positive and strictly increasing on $(0, 1]$ for $i = 2, 3, \dots, 2k + 1$, $\max\{y_{2i+1}(x) : 0 \leq x \leq 1/2\} > b_i$ for $i = 1, 2, \dots, k$, and

$$\begin{aligned} y_i(1) &< c_\ell, & \text{if } i \leq 2\ell + 1, \\ y_i(1) &> c_\ell, & \text{if } i > 2\ell + 1. \end{aligned}$$

Proof. First, extend f to all of \mathbb{R} by defining $f(y) = f(c_k)$ for $y > c_k$ and $f(y) = f(0)$ for $y < 0$. The proof merely assembles the results of Lemmas 4 and 5 and is quite similar to the corresponding proof in [2]. The idea is to alternately apply Lemmas 4 and 5. Lemma 4 with $c = c_0$ gives the solution $y_1 = y_{s_1}$ and the value $m_1 = m^*$, with $y_{m_1}^{(j)}(1) > 0$. Then Lemma 5 with $b = b_1$ gives the value $m_2 = m'$, with $y_{m_2}^{(j)}(1) < 0$. We now verify that the $m_2 > m_1$. Clearly $m_2 \neq m_1$, so supposing that $m_2 < m_1$, Lemma 5 implies $\|y_{m_1}\| > b_1$, but Lemma 4 gives the contradiction $\|y_{m_1}\| < c_0 < b_1$. Next we apply Lemma 4 again with $a = c_1$ and get $m_3 = m^*$, with $y_{m_3}^{(j)}(1) > 0$. We next verify that $m_3 > m_2$. Since clearly $m_3 \neq m_2$, we suppose that $m_3 < m_2$. Then by Lemma 5, $\|y_{m_2}\| \leq qb_1 < c_1$, but the final line of Lemma 4 implies $y_{m_2}^{(j)}(1) > 0$, again a contradiction. Continuing in this way, we get $0 < m_1 < m_2 < \dots < m_{2k+1}$ for which $y_{m_i}^{(j)}(1) > 0$ if i is odd and of the opposite sign when i is even. Thus continuous dependence gives the solutions $y_2, y_3, \dots, y_{2k+1}$ we desire. For $i > 1$, we may apply Lemma 2 with $w = y_i^{(j)}$ on $[0, 1]$ to conclude that $y_i^{(j)}(x) > 0$ on $(0, 1)$. Repeated integration, if necessary, then shows that $y_i'(x) > 0$ and $y_i(x) > 0$ on $(0, 1)$. The remaining statements, with the exception of the final statement, are immediate from Lemmas 4 and 5. The final statement is an implication of the final line in Lemma 4: since $y_{2\ell+1}^{(n-1)}(0)$ is the largest value of m such that $y_{2\ell+1}^{(j)}(1) = 0$ and $y_{2\ell+1}(1) \leq c_{2\ell+1}$, then for any values of $m > y_{2\ell+1}^{(n-1)}(0)$ for which y_m solves the boundary value problem, it follows that $y_m(1) > c_{2\ell+1}$. \square

If $f(0) = 0$, then the trivial solution qualifies as the y_1 of Theorem 2 and it may be that no other solution qualifies as y_1 . If $f(0) > 0$, it is easy to see that there exists b_0 with $0 < b_0 < qb_0 < c_0$ so that $f(y) \geq Kb_0$ for $b_0 \leq y \leq qb_0$. The next theorem says that this condition is sufficient to guarantee that y_1 may be chosen positive in Theorem 2.

Theorem 3. *In addition to the hypotheses of Theorem 2, suppose that there exists b_0 with $0 < b_0 < qb_0 < c_0$ for which $f(y) \geq Kb_0$ for $b_0 \leq y \leq qb_0$. Then the y_1 in Theorem 2 may be chosen strictly positive and strictly increasing on $(0, 1)$.*

Proof. One only needs to begin the proof of Theorem 2 by using Lemma 5 with $b = b_0$ to get $m_0 = m'$. Then when Lemma 4 is applied with $c = c_0$, one gets $m_1 = m^* > m_0$ and $s_1 \in (m_0, m_1)$ so that $y_1 = y_{s_1}$. \square

With one extra hypothesis in Theorem 2, we can get one more positive solution as follows.

Theorem 4. *In addition to the hypotheses of Theorem 2, suppose that $f : \mathbb{R} \rightarrow [0, \infty)$ and there exists $b_{k+1} > c_k$ so that $f(y) \geq Kb_{k+1}$ for $b_{k+1} \leq y \leq qb_{k+1}$. Then, in addition to the solutions guaranteed by Theorem 2, there exists an additional solution y_{2k+2} with $y_{2k+1}^{(j)}(0) > y_{2k}^{(j)}(0)$ and $\|y_{2k+2}\| \leq qb_{k+1}$.*

Proof. In the proof of Theorem 2, one just applies Lemma 5 one additional time with $b = b_{k+1}$ to get the one additional solution. \square

In Table 1, we provide values of the numbers L, K, q for $3 \leq n \leq 6$ and corresponding values of j . The careful reader will know that the values of K and q are linked, but are independent of L . For fixed j , the values of L increase with n ; for fixed n , the values of L decrease with j . On the other hand, the values of both K and q increase with n for fixed j , but for fixed n , as j increases, K decreases while q increases. We expect that this trend persists for larger values of n .

The calculation of the numbers in Table 1 is fairly easy since we have explicit formulas for each of these numbers. Our theorem does not cover the case $j = 0$ considered by Henderson and Thompson [6]. Because of the error in [6] in the case $j = n - 2$, no comparison is possible.

Table 1
Values of L, K, q for various pairs n, j

n	$j = 1$	$j = 2$	$j = 3$	$j = 4$
3	12, 64, 8/3			
4	72, 1152, 5	24, 384, 7		
5	480, 24576, 48/5	180, 9216, 68/5	80, 3072, 76/5	
6	3600, 614400, 56/3	1440, 245760, 80/3	720, 92160, 30	360, 30720, 94/3

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